Relating Formulas to Taclets

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This note relates the formulas - axioms, definitions, lemmas- used in the technical report [1] to the names of the taclets that implement them in the KeY system. The so far unpublished technical report [1] is a slight extension of the unrestrictedly available [2].

A note on notation

in [1]	Tableau paper	taclets
$f^{m,n}(x)$	oHNf(n,m) for $f^{m,n}(n+1)$	`oHNf(n,m)` for $f^{m,n}(n+1)$
$o(n,m)$	oGS(n,m)	`oGS(n,m)`

The general three-argument function $f^{n,m}(x)$ has no counterpart neither in the Tableaux paper nor in KeY's taclets.

1 **Core Axioms** from [1, Figure 2]

1. $\forall x, y, z(x < y \wedge y < z \rightarrow x < z)$ — transitivity
   `olt-transAxiom`, `olt_transAut`
2. $\forall x(\neg x < x)$ — strict order
   `olt_irref_Axiom`
3. $\forall x, y(x < y \vee x \doteq y \vee y < x)$ — total order
   `olt_total_Axiom`
4. $\forall x(0 \leq x)$ — 0 is smallest element
   `oleq_zeroAxiom`
5. $0 < \omega \wedge \neg\exists x(\omega \doteq x + 1)$ — ω is a limit ordinal
   `omegaDef1`
6. $\forall y(0 < y \wedge \forall x(x < \omega \rightarrow x + 1 < y) -> \omega \leq y)$
   `omegaDefLeastInf` — ω is the least limit ordinal
7. $\forall x(x < x + 1) \wedge \forall x, y(x < y \rightarrow x + 1 \leq y)$ — successor function
   `oSucc`, `oLeastSucc`
8. $\forall z(z < \alpha \rightarrow t[z/\lambda] \leq sup_{\lambda<\alpha} t)$
   `osupDef` — def of supremum, part 1
9. $\forall x(\forall z(z < \alpha \rightarrow t[z/\lambda] \leq x) \rightarrow sup_{\lambda<\alpha} t \leq x)$ — def of supremum, part 2
   `osupDef`
10. $\forall x(\forall y(y < x \rightarrow \phi(y)) \rightarrow \phi) \rightarrow \forall x\phi$ — transfinite induction scheme
    `oIndBasic`, `oInd`

2 **Definitional Extension** from [2, Figure 5]

$\forall x, y(x \leq y \leftrightarrow x \doteq y \vee x < y)$ (less or equal relation)
`oleq_Def`

$\forall x(lim(x) \leftrightarrow 0 < x \wedge \neg\exists y(y + 1 \doteq x))$ (limit ordinal)
`olimDef`

$\forall x, y(max(x, y) \doteq \text{if } x \leq y \text{ then } y \text{ else } x)$ (maximum operator)
`omaxDef`

3 **Derivable Lemmas** from [1, Figure 6]

1. $\exists x\phi \rightarrow \exists x(\phi \wedge \forall y(y < x \rightarrow \neg\phi[y/x]))$
   `least_number_principle`
2. $\forall x, y, z(x \leq y \wedge y \leq z \rightarrow x \leq z)$
   `oleq_trans`, `oleq_transAut`
3. $\forall x, y, z(x \leq y \wedge y < z \rightarrow x < z)$
   `oltleq_trans`, `oltleq_transAut`
4. $\forall x, y, z(x < y \wedge y \leq z \rightarrow x < z)$
   `oleqolt_trans`, `oleqolt_transAut`
5. $\forall x, y(x \leq y \wedge y \leq x \rightarrow x \doteq y)$
   `oleq_antisym`
6. $\forall x, y, z(max(x, y) < z \leftrightarrow (x < z \wedge y < z))$
   `omaxGreater`
7. $\forall x, y, z(z < (max(x, y) \leftrightarrow (z < x \vee z < y)))$
   `omaxLess`
8. $\forall x, y, z(max(x, y) \leq z \leftrightarrow (x \leq z \wedge y \leq z))$
   `omaxGeq`
9. $\forall x, y, z(z \leq (max(x, y) \leftrightarrow (z \leq x \vee z \leq y)))$
   `omaxLeq`
10. $\forall x(max(0, x) \doteq max(x, 0) \doteq x)$
    `omax0Lef` and `omax0Right`
11. $sup_{\lambda<0}\ t \doteq 0$
    `osup0`
12. $sup_{\lambda<1}\ t \doteq t[0/\lambda]$
    `osup1`
13. $\forall x(lim(x) \rightarrow sup_{\lambda<x}\ \lambda \doteq x)$
    `oselfSup`
14. $\forall x(sup_{\lambda<x+1}\ t \doteq max(sup_{\lambda<x}\ t, t[x/\lambda]))$
    `osupSucc`
15. $\forall x(\forall y(y < x \rightarrow t_1[y/\lambda] \doteq t_2[y/\lambda]) \rightarrow sup_{\lambda<x}\ t_1 \doteq sup_{\lambda<x}\ t_2)$
    `osupEqualTerms`
16. $\forall \alpha_1, \alpha_2($
    $\forall x(x < \alpha_1 \rightarrow \exists y(y < \alpha_2 \wedge t_1[x/\lambda] \leq t_2[y/\lambda]))\wedge$
    $\forall y(y < \alpha_2 \rightarrow \exists x(x < \alpha_1 \wedge t_2[y/\lambda] \leq t_1[x/\lambda]))$
    $\leftrightarrow sup_{\lambda<\alpha_1}\ t_1 \doteq sup_{\lambda<\alpha_2}\ t_2)$
    `osupMutualCofinal`

17. $lim(\lambda) \leftrightarrow \lambda \neq 0 \wedge \forall ov(ov < \lambda \rightarrow (ov + 1) < \lambda$
    `olimDefEquiv`
18. $\forall\lambda(t_1 \leq t_2 \rightarrow sup_{\lambda<b}\ t_1 \leq sup_{\lambda<b}\ t_2$
    `osupLocalLess`

4 **Definitional Extension** from [1, Figure 7]

$\forall x(x + 0 \doteq x)$
`oadd_Def0Right`

$\forall x, y(x + (y + 1) \doteq (x + y) + 1)$
`oadd_DefSucc`

$\forall x, y(lim(y) \rightarrow x + sup_{\lambda<y}\lambda \doteq sup_{\lambda<y}(x + \lambda))$
`oadd_DefLim`

$\forall x(x * 0 \doteq 0)$
`otimes_Def0Righ`

$\forall x, y(x * (y + 1) \doteq (x * y) + x)$
`otimes_DefSucc`

$\forall x, y(lim(y) \rightarrow x * y \doteq sup_{\lambda<y}(x * \lambda))$
`otimes_DefLi`

$\forall x(x^0 \doteq 1)$
`oexp_Def0Righ`

$\forall x, y(x^{y+1}) \doteq (x^y) * x)$
`oexp_DefSucc`

$\forall x, y(lim(y) \wedge x \neq 0 \rightarrow x^y \doteq sup_{\lambda<y}(x^\lambda))$
`oexp_DefLim`

$\forall y(lim(y) \rightarrow 0^y \doteq 0)$
`oexp_DefLim0`

5 **Derivable Lemmas** from [1, Figure 8]

1. $\forall x, y(y \neq 0 \rightarrow x < x + y)$
   `oaddStrictMonotone`
2. $\forall x, y(x \leq x + y)$
   `oaddMonotone`
3. $\forall x, y(y \leq x + y)$
   `oaddLeftMonotone`
4. $\forall x, y, z(x < y \rightarrow z + x < z + y)$
   `oltAddLessLeft`
5. $\forall x, y, z(x \leq y \rightarrow x + z \leq y + z)$
   `oleqAddLessRight`

6. $\forall x, y, z(x + y < x + z \to y < z)$
`oAddOltPreserv`
7. $\forall x, y, u, w(x < y \wedge u < w \to x + u < y + w)$
`oadd2olt`
8. $\forall x, y, u, w(x \leq y \wedge u \leq w \to x + u \leq y + w)$
`oadd2oleq`
9. $(i < \omega \wedge j < \omega) \to i + j < \omega$
`oaddLessOmega`

6 Derivable Lemmas on Addition from [1, Figure 9]

1. $\forall x(0 + x \doteq x)$
`oadd0Left`
2. $\forall x, y(x + y \doteq 0 \leftrightarrow x \doteq 0 \wedge y \doteq 0)$
`ozerosum`
3. $\forall x, y, z(max(z + x, z + y) \doteq z + max(x, y)$
`omaxAddL`
4. $\forall x, y, z(max(x + z, y + z) \doteq max(x, y) + z$
`omaxAddR`
5. $\forall x(x < \omega \to x + \omega \doteq \omega)$
`oaddLeftomega`
6. $\forall x(lim(\lambda) \to lim(x + \lambda))$
`olimAddolim`
7. $\forall x(\omega \leq x \to \exists \lambda, n(lim(\lambda) \wedge n < \omega \wedge x \doteq \lambda + n))$
`repLimPlusNat`
8. $\forall x, y(x \leq y \to \exists z(x \doteq y + z))$
`ordDiff`
9. $\forall x, y, z(x + (y + z) \doteq (x + y) + z)$
`oaddAssoc`
10. $\neg \exists x(x + 1 \doteq 0)$
`o0notSucc`
11. $\forall x, y(x < y \to (x + 1) < (y + 1))$
`oltPlusOne`
12. $\forall x, y((x + 1) \doteq (y + 1) \to x \doteq y)$
`oAddOneInj`
13. $\forall x, y, z((z + x) \doteq (z + y) \to x \doteq y)$
`oaddRightInjective`
14. $sup_{\lambda < x}\ (i + t) \doteq i + sup_{\lambda < x}\ t$ if λ does not occur in i and $x > 0$
`osupAddStaticTerm`
15. $i + j \doteq j$ if $\omega \leq j$ and $i < \omega$
`oaddLeftomega`, `oaddLeftAbsorb`

7 Derivable Lemmas on Multiplication from [1, Figure 10]

1. $\forall x(1 * x \doteq x * 1 \doteq x)$
`otimesOneRight` and `otimesOneLef`

2. $\forall x(0 * x \doteq 0)$
   `otimesZeroLeft`
3. $\forall x, y, z((0 < z \wedge x < y) \rightarrow z * x < z * y)$
   `otimesMonotoneQ`, `otimesMonotone`
4. $\forall x, y, z(z * x < z * y) \rightarrow (0 < z \wedge x < y))$
   `otimesMonotoneRev`
5. $\forall x, y, z(0 < z \wedge z * x \doteq z * y \rightarrow x \doteq y)$
   `otimesLeftInjective`
6. $\forall x, y, z(x \leq y \rightarrow x * z \leq y * z)$
   `otimesLeftMonotone`
7. $\forall x, y(x \neq 0 \rightarrow y \leq x * y$
   `otimesRightMonotoneQ`
8. $i * j \doteq 0 \leftrightarrow i \doteq 0 \vee j \doteq 0$
   `otimesZero`
9. $i * j \doteq 1 \leftrightarrow i \doteq 1 \wedge j \doteq 1$
   `otimesOne`
10. $\forall x, y(x < \omega \wedge y < \omega \rightarrow x * y < \omega)$
    `otimesFinite`, `otimesFiniteAxiom`
11. $\forall x(0 < x < \omega \rightarrow x * \omega \doteq \omega)$
    `otimesNomegaQ`, `otimesNomega`
12. $\forall x, y, z(max(z * x, z * y) \doteq z * max(x, y))$
    `omaxTimesL`
13. $\forall x, y, z(max(x * z, y * z) \doteq max(x, y) * z)$
    `omaxTimesR`
14. $sup_{\lambda < x}\ (i * t) \doteq i * sup_{\lambda < x}\ t$
    `osupTimesStaticTerm`
15. $\forall i, j, k(i * (j + k) \doteq i * j + i * k)$
    `odistributiveQ`
16. $\forall i, j, k((i * j) * k \doteq i * (j * k)$
    `otimesAssocQ`
17. $\forall \lambda, n, i(lim(\lambda) \wedge n < \omega \rightarrow i * \lambda \doteq (i + n) * \lambda)$
    `Klaua26c1a`
18. $\forall \lambda, x((lim(\lambda) \wedge 0 < x < \omega) \rightarrow x * \lambda \doteq \lambda)$
    `otimesNlimit`
19. $\forall i, j(1 < i \wedge 1 < j \rightarrow i + j \leq i * j)$
    `oleqAddTimes`
20. $\forall i, \lambda(0 < i \wedge lim(\lambda) \rightarrow lim(i * \lambda))$
    `olimtimes1Q`, `olimtimes1`
21. $\forall i, \lambda(0 < i \wedge lim(\lambda) \rightarrow lim(\lambda * i))$
    `olimtimes2Q`, `olimtimes2`

8 Derivable Lemmas on Finite Ordinals from [1, Figure 11]

1. $\forall x, y(x < \omega \wedge y < \omega \rightarrow x + y \doteq y + x)$
   `oaddFiniteCom`

2. $\forall i, j, k((i < \omega \wedge j < \omega \wedge k > \omega) \rightarrow (i + j) * k \doteq i * k + j * k)$
   `odistributiveFinite`
3. $\forall x, y(x < \omega \wedge y < \omega \rightarrow x * y \doteq y * x)$
   `otimesFiniteCom`

9 Derivable Lemmas on Exponentiation from [1, Figure 12]

1. $x^1 \doteq x$
   `oexpOne`
2. $\forall x(0 < x \rightarrow 0^x \doteq 0)$
   `oexpZeroBase`
3. $\forall x(1^x \doteq 1)$
   `oexpOneBase`
4. $\forall x, y(1 < x \wedge 1 < y \rightarrow x < x^y)$
   `oexpLeftIncreasing`
5. $\forall x, 0 < y \rightarrow x \leq x^y)$
   `oexpLeftWeakIncreasing`
6. $\forall x, y(1 < x \wedge 0 < y \rightarrow 1 < x^y)$
   `:oexpGreaterOne`
7. $\forall x, y(1 < x \rightarrow 1 \leq x^y)$
   `oexpGreaterEqualOne`
8. $\forall x, y(x \neq 1 \rightarrow x * y \leq x^y)$
   `oexpGreatertimes`
9. $\forall x, y(1 < x \rightarrow y \leq x^y)$
   `oexpRightNondecreasing`
10. $\forall x, y_1, y_2(1 < x \wedge y_1 < y_2 \rightarrow x^{y_1} < x^{y_2})$
    `oexpRightMonotoneQ`
11. $\forall x, y_1, y_2(1 < x \wedge x^{y_1} < x^{y_2} \rightarrow y_1 < y_2)$
    `oexpRightMonotoneRevQ`
12. $\forall x_1, x_2, y(x_1 < x_2 \rightarrow x_1^y \leq x_2^y)$
    `oexpLeftMonotoneQ`
13. $\forall x_1, x_2, y(x_1 < x_2 \wedge 0 < y \wedge \neg lim(y) \rightarrow x_1^y < x_2^y)$
    `oexpLeftSuccessorMonotoneQ`
14. $x^y \doteq 0 \rightarrow x \doteq 0 \wedge y \neq 0$
    `oexpEqualsZero`
15. $x^y \doteq 1 \rightarrow y \doteq 0 \vee x \doteq 1$
    `oexpEqualsOne`
16. $\forall x, y(x < \omega \wedge y < \omega \rightarrow x^y < \omega)$
    `oexpFinite`
17. $\forall x, y(1 < x \wedge x < \omega \rightarrow x^\omega \doteq \omega)$
    `oexpNOmega`
18. $\forall x, y((0 < x \wedge lim(y) \rightarrow lim(y^x)$
    `olimexp1limQ`
19. $\forall x, y(1 < x \wedge lim(y) \rightarrow lim(x^y)$
    `olimexp2limQ`, `olimexp2lim`

20. $\forall x, y, z(x^{y+z} \doteq x^y * x^z)$
`oexpDistr`
21. $\forall x, y, z((x^y)^z \doteq x^{y*z})$
`oexpTriple`
22. $\forall b((0 < b \wedge \forall x(x < b \rightarrow 0 < j)) \rightarrow sup_{x<b}(i^j) = i^{sup_{x<b}(j)})$
for all terms i, j such that x does not occur in i.
`osupExpStaticTerm`

10 **Positive Integers as Ordinal** from [1, Figure 14]

Definitional Extension

1. $onat(0) \doteq 0$
`onatZeroDef`
2. $\forall n(0 \leq n \rightarrow onat(n+1) \doteq onat(n) + 1)$
`onatSuccDef`

Derived Lemmas

3. $onat(1) \doteq 1$
`onatOne`
4. $\forall n, m(0 \leq n \wedge 0 \leq m \rightarrow onat(n+m) \doteq onat(n) + onat(m)$
`onatoadd`
5. $\forall n, m((0 \leq n \wedge 0 \leq m) \rightarrow (onat(n) \doteq onat(m) \rightarrow n \doteq m))$
`onatInj`
6. $\forall n, m((0 \leq n \wedge 0 \leq m) \rightarrow (onat(n) < onat(m) \leftrightarrow n < m))$
`onatolt`, `onatoltAut`
7. $\forall n(0 \leq n \rightarrow onat(n) < \omega)$
`onatLessOmega`
8. $\forall i_1, i_2, j_1, j_2 \; ((0 \leq i_1 \wedge 0 \leq i_2 \wedge 0 \leq j_1 \wedge 0 \leq j_2) \rightarrow$
$\omega * onat(i_1) + onat(j_1) < \omega * onat(i_2) + onat(j_2)$
$\leftrightarrow i_1 < i_2 \vee (i_1 \doteq i_2 \wedge j_1 < j_2))$
`oltlexicographic`

11 Definitional extension for termination proof from [1, Figure 19]

$\forall n \forall m(2 <= n \wedge 0 <= m \wedge m < n \rightarrow o(n,m) = onat(m))$
`oGSDef1`

$\forall n, m, k, a, c($
$m = (n^k * a) + c \wedge 2 <= n \wedge 1 <= k \wedge 0 < a < n \wedge c < n^k$
$\rightarrow o(n,m) = \omega^{o(n,k)} * onat(a) + o(n,c))$
`oGSDef2`

$\forall base, m(2 <= base \wedge 0 <= m \wedge m < base \rightarrow \mathrm{oHNf}(base, m) = m)$
`oHNfDef1`

$\forall base, m, k, a, c($
$m = (base^k * a) + c \wedge 2 <= base \wedge 1 <= k \wedge 0 < a \wedge a < base \wedge c < base^k$
$\rightarrow \mathrm{oHNf}(base, m) = (base + 1)^{\mathrm{oHNf}(base,k)} * a + \mathrm{oHNf}(base, c))$
`oHNfDef2`

12 Derivable lemmas on $o(n, m)$ from [1, Figure 20]

1. $\forall n(2 \leq n \rightarrow o(n, 0) = onat(0))$
   `oGSZero`
2. $\forall n, e(1 < n \wedge 0 < e \rightarrow 0 < o(n, e))$
   `oGSGreaterZeroQ`
3. $\forall n, e(1 < n \wedge 0 \leq e \rightarrow 0 \leq o(n, e))$
   `oGSGEZero`
4. $\forall n, m_1, m_2((2 \leq n \wedge 0 \leq m_1 \wedge m_1 < m_2) \rightarrow o(n, m_1) < o(n, m_2))$
   `oGSstrictMonotone`

13 Derivable lemmas on $\mathrm{oHNf}(m, n)$ $(=f^{n,m}(m+1))$ from [1, Figure 21]

1. $\forall n(2 \leq n \rightarrow \mathrm{oHNf}(n, 0) = 0)$
   `oHNfZero`
2. $\forall base, m(2 \leq base \wedge base \leq m \rightarrow m < \mathrm{oHNf}(base, m))$
   `oHNfIncreasing`
3. $\forall base, m(2 \leq base \wedge 0 \leq m \rightarrow m \leq \mathrm{oHNf}(base, m))$
   `oHNfWeakIncreasing`
4. $\forall base, m, k(m = base^k \wedge 2 \leq base \wedge 1 \leq k \rightarrow \mathrm{oHNf}(base, m) = (base + 1)^{\mathrm{oHNf}(base,k)})$
   `oHNfDef2a`
5. $\forall m_2, m_1, base; (2 \leq base \wedge 0 \leq m_1 \wedge m_1 < m_2 \wedge \mathrm{oHNf}(base, m_1) < \mathrm{oHNf}(base, m_2))$
   `oHNfMonotone`

6. $\forall base, k, a, c($
   $2 \leq base \wedge 1 \leq k \wedge 0 < a \wedge a < base \wedge c < base^k \wedge 0 \leq c$
   $\rightarrow \mathrm{oHNf}(base, c) < (base + 1)^{\mathrm{oHNf}(base,k)})$
   `oHNfcbLemma`
7. $\forall base, m(2 \leq base \wedge 0 \leq m \rightarrow o(base, m) = o(base + 1, \mathrm{oHNf}(base, m)))$
   `oGSnextb`

References

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2. P. H. Schmitt. A first-order theory of ordinals. Technical Report 6, Department of Informatics, Karlsruhe Institute of Technology, 2017.